Complex networks are an emerging property of hierarchical preferential attachment

Laurent Hébert-Dufresne, Edward Laurence, Antoine Allard, Jean-Gabriel Young and Louis J. Dubé

Département de Physique, de Génie Physique et d'Optique, Université Laval, Québec, Canada

Summary

Scale independence is observed in all aspects of human life and often modeled through preferential attachment (PA). Network science and PA processes tend to focus on one feature at a time; e.g. degree distribution [1] or community structure [2].

Complex networks are constructs obtained by projecting complex hierarchical systems on a set of nodes and links; collapsing geographical/age/cultural/professional correlations.

Why not directly model the hierarchical system itself instead of its projection?
What can emerge from a simple hierarchy of scale independent organizations?

Hierarchical Preferential Attachment features

- the simplicity of preferential attachment,
- complex networks as an emerging property.

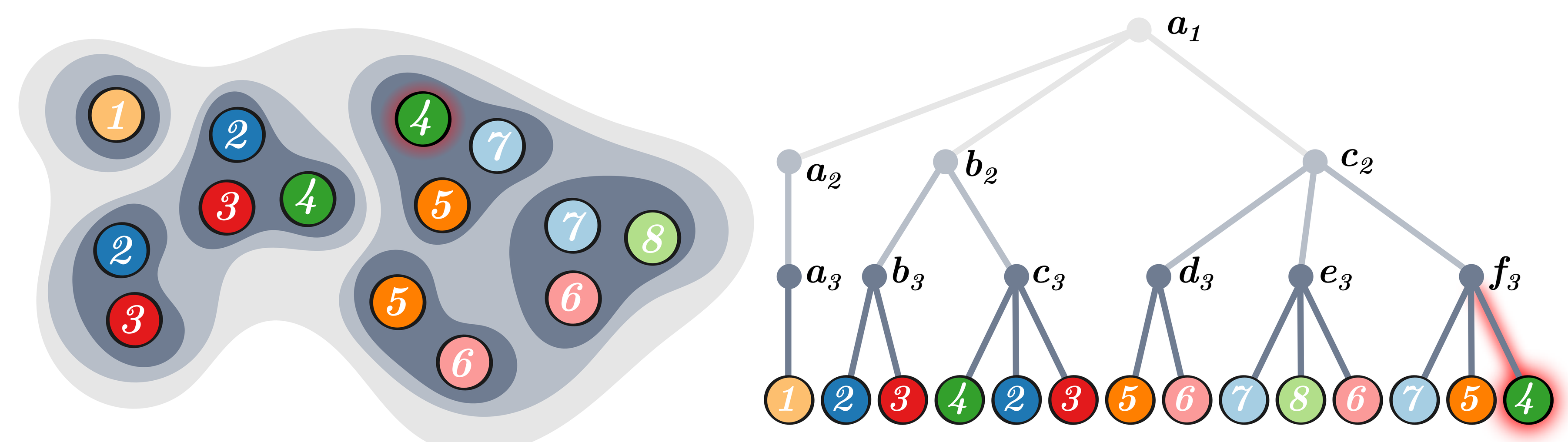
Complex networks emerge from hierarchy?

Hierarchical systems produce networks when projecting under a chosen level of structure. Correlations inter and intra levels of structures dictate properties of the network:

- locally: degree and clustering;
- globally: centrality, self-similarity;
- + complex properties such as geometrical mapping!

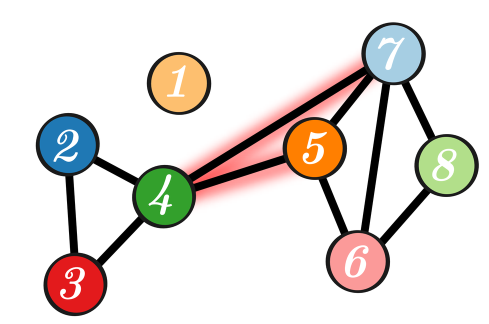
Hierarchy makes complex networks complex.
HPA is perfectly suited to model scale-independent networks.

Hierarchical Preferential Attachment (HPA)



HPA ≡ colored balls are thrown in embedded bins.

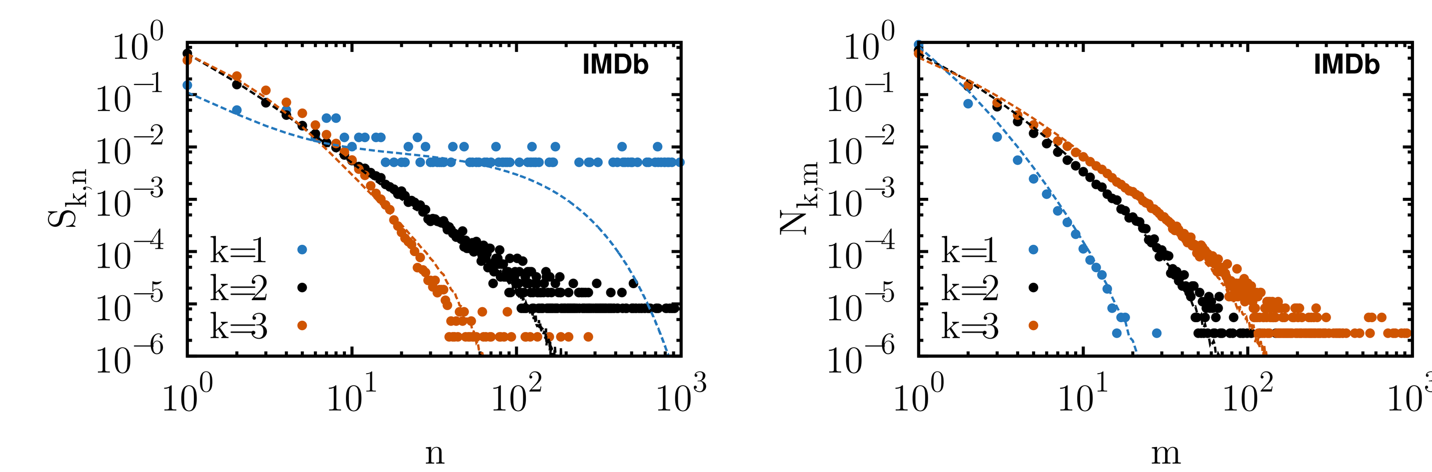
- Embedded bins (top left) represent a tree-like hierarchy (top right).
- Balls/bins are different structural levels (e.g. people, communities, cities, countries).
- At level i, let $p_i \equiv$ probability that a ball falls in a new bin;
  $q_i \equiv$ probability that the color of the ball is new for that bin.
- Whenever an existing bin and/or color has to be chosen, it is done preferentially to its size/frequency at that level.
- A network is obtained by projecting the system on a given level.
- For the network on the right: colors found in a common bin on the lowest levels are linked in the network. This could be a network of collaborating scientists, projecting labs (level 3) across cities (level 2) and countries (level 1) on a single social network.
- Other projections are possible; e.g., a network of the boxes of level 3 that share at least one color could be a network of collaborations between research groups.



Case study: movie production structure

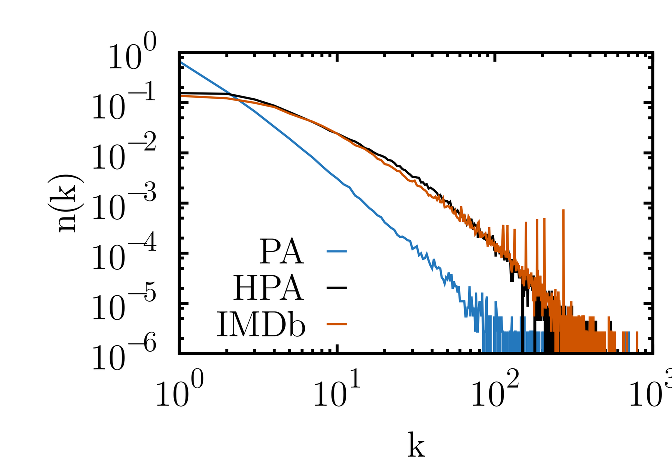
Hierarchy: **countries** (largest bins, level $k=1$) containing **production companies** (middle bins, level $k=2$) producing **movies** (smallest bins, level $k=3$) with **producers** (colored balls).

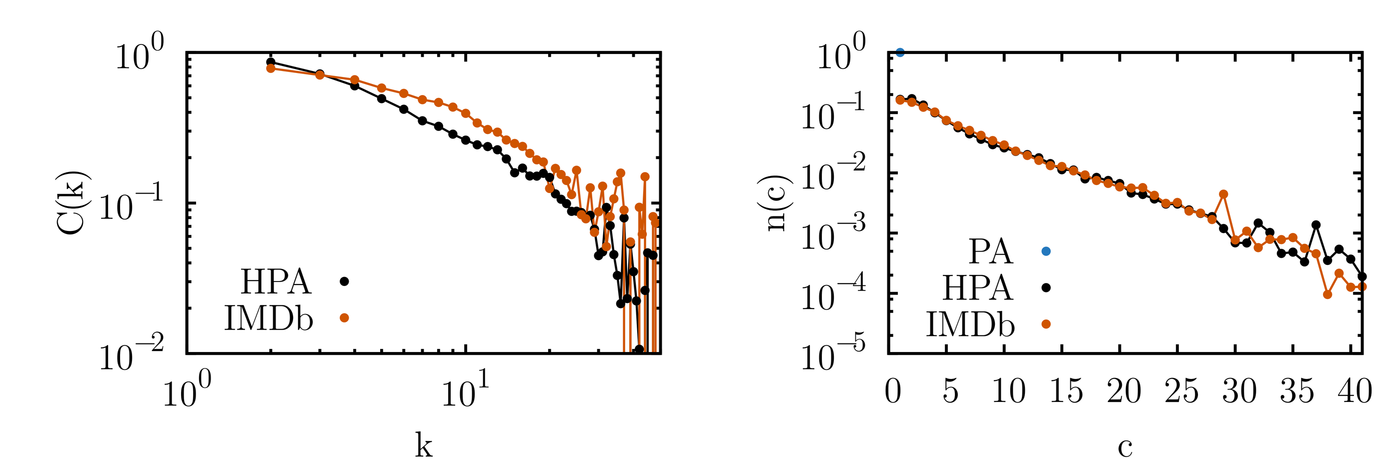
We set all $\{p_i, q_i\}$ with $S_{k,n}$ (distribution of level k structures of sizes n) and $N_{k,m}$ (distribution of colors appearing in m level k structures) by comparing data (dots) and simulations (lines).



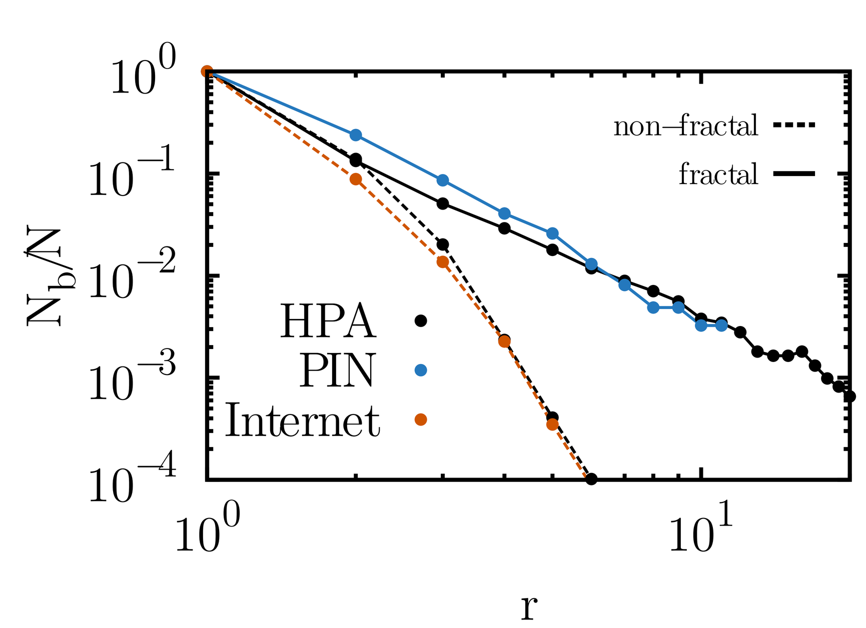
Projection for a realization of HPA:

- Project the system in a network of co-producing credits: links between producers who have produced together, regardless of companies and country.
- Random **HPA network** captures structure from real network not captured by Standard PA:
  1. degree distribution $n(k)$
  2. local clustering coefficient $C(k)$ around nodes of degree k ($C(k) = 0 \; \forall \; k$ in Standard PA)
  3. distribution $n(c)$ of coreness c, i.e. number of nodes in a shell of the k-core decomposition ($n(c) = \delta_{c,1}$ in Standard PA)





Proof of concept: Fractality and geometrical mapping



Fractal (& non-fractal) networks from hierarchy:

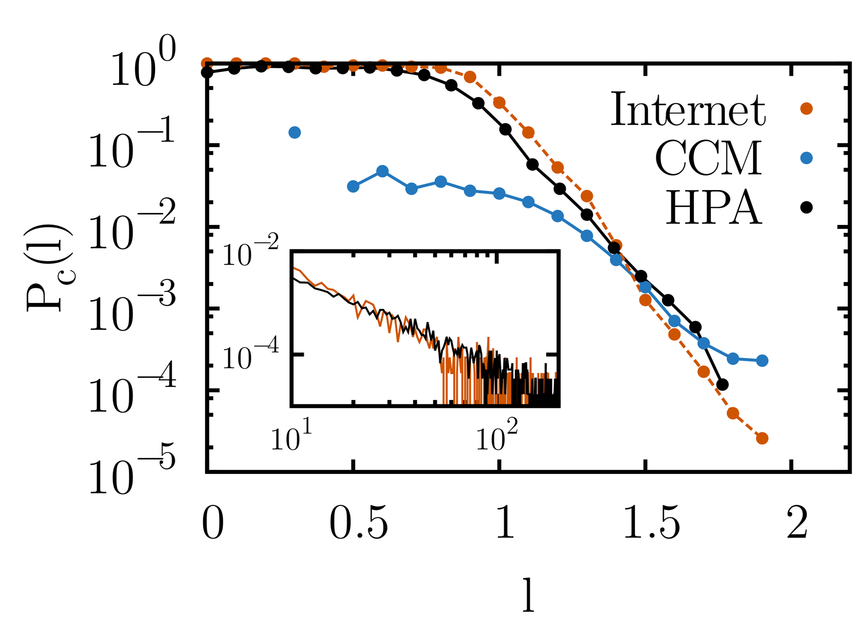
HPA yields fractal and non-fractal networks: self-similarity might imply hierarchy, the opposite is not true.

- Well-mixed hierarchies have a network diameter D scaling with the logarithm of the number of nodes N (non-fractal)
- Systems with well defined hierarchy lead to a power-law relation between D and N (fractal)

Fractality is uncovered with box-counting [3]: groups of nodes within a distance r (number of links) are assigned to the same box. The fractal dimension d_b relates the number N_b of boxes and their size r: $N_b \propto r^{-d_b}$.

Figure on the left: box counting results on a fractal network (protein interaction network of Homo Sapiens) and a non-fractal network (the Internet at the level of autonomous systems) [3].

∴ HPA models how both of these networks span and cover their respective space.



Hyperbolic mapping of networks [4]:

Mapping of a network: assign geometrical positions to nodes to embed the network in an hyperbolic space. Nodes close (in links) in the network must be geometrically close (in space).

Navigability of complex networks:

- predicts existence of links as a function of geometrical distance between nodes, enabling an efficient navigation.
- is not captured by classical preferential attachment.

Figure on the left: probability of connection $P_c(l)$ between nodes at a distance l after an inferred projection of the networks unto an hyperbolic space [4].

- The Internet and its HPA model share a similar scaling exponent for their degree distribution (inset).
- The CCM (Correlated Configuration Model) is a rewired Internet preserving degree distribution and degree-degree correlations, but obviously lacking the more complex structural correlations.

∴ Geometrical constraints can emerge simply from hierarchy.

Bibliography and Acknowledgements

HPA is presented in: L. Hébert-Dufresne *et al.*, arXiv:1312.0171 (2013)

[1] A.-L. Barabási & R. Albert, Science 286 (1999)
[2] L. Hébert-Dufresne *et al.*, Phys. Rev. Lett. 107 (2011)
[3] C. Song, S. Havlin & H. A. Makse, Nature 433 (2005)
[4] F. Papadopoulos *et al.*, Nature 489 (2012)

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